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The algebraic Bethe ansatz for open $A_{2n}^{(2)}$ vertex model

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Abstract: We solve the $A_{2n}^{(2)}$ vertex model with all kinds of diagonal reflecting matrices by using the algebraic Bethe ansatz, which includes constructing the multi-particle states and achieving the eigenvalue of the transfer matrix and corresponding Bethe ansatz equations. When the model is $U_q(B_n)$ quantum invariant, our conclusion agrees with that obtained by analytic Bethe ansatz method.

Keywords: Bethe Ansatz.

Contents

1. Introduction

In solving integrable models, one of powerful tools is the analytical Bethe ansatz method which was proposed by Reshetikhin for close chains [1], and was generalized into quantum-algebra-invariant open chains [2]. Then it was employed to solve a family of both quantum-algebra-invariant and non-quantum-algebra-invariant open chains [3]–[5]. For the result is far from rigorous, one more satisfactory approach would be the algebraic Bethe ansatz (ABA), by which the Bethe ansatz equations and the eigenstates can be obtained.

Recently, the algebraic Bethe ansatz [6]–[8] has been developed by Martins [9]–[11] for a large family of vertex models with periodic boundary. Its generalizations have been applied into the vertex models with open boundary conditions in refs. [12]–[15] and refs. [16]–[18]. All of these show that the ABA could be suit for the system with higher rank algebra symmetry. Although some open boundary vertex models, such as $A_{2n}^{(2)}$ model [19, 20], have been solved by the analytical Bethe ansatz or other methods, it is still worthy to reconsider those by ABA. Here we will formulate the algebraic Bethe ansatz solution for the $A_{2n}^{(2)}$ vertex model with diagonal reflecting matrices.

The $A_{2n}^{(2)}$ model at the case of $n = 1$ is also called Izergin-Korepin model [21] which, under the open boundary conditions, can be related to the loop models [5] and flexible self- avoiding polymer chain [22]. When $n > 1$, the model with trivial reflecting matrices was solved by the analytical Bethe ansatz. However, for non-trivial reflecting matrix, the exact solutions remain unknown. In this paper, we expect to solve the model with all kinds of diagonal reflecting matrices by using the algebraic Bethe ansatz, which includes constructing the multi-particle states and achieving the eigenvalue of the transfer matrix and corresponding Bethe ansatz equations. When the model is $U_q(B_n)$ quantum invariant, our conclusion agrees with that obtained by analytic Bethe ansatz method [3].

The present paper is organized as following. In section 2 we introduce $A_{2n}^{(2)}$ vertex model and list all diagonal $K_\pm$ matrices governing the boundary terms in the hamiltonian. Section 3 devotes to construct m-particle eigenfunctions and to derive out the eigenvalue and the Bethe ansatz equations. A brief summary and discussion about our main result are included in section 4. Some necessary calculations and coefficients are given as the appendices A and B.

2. The vertex model and integrable boundary conditions

The R matrix for the $A_{2n}^{(2)}$ model used here is [3]

$$\begin{aligned}R^{(n)}(u) = a_n(u)\sum_{i\neq\bar{i}} E_{ii}\otimes E_{ii} + b_n(u)\sum_{i\neq j,\bar{j}} E_{ii}\otimes E_{jj} + \left(\sum_{i<\bar{i}} c_n(u,i) + \sum_{i>\bar{i}} \bar{c}_n(u,i)\right) E_{\bar{i}\bar{i}}\otimes E_{\bar{i}i} + \\ + \left(\sum_{i<j,j\neq\bar{i}} d_n(u,i,j) + \sum_{i>j,j\neq\bar{i}} \bar{d}_n(u,i,j)\right) E_{ij}\otimes E_{\bar{i}\bar{j}} + e_n(u)\sum_{i\neq\bar{i}} E_{ii}\otimes E_{\bar{i}\bar{i}} + \\ + f_n(u) E_{n+1n+1}\otimes E_{n+1n+1} + \left(g_n(u)\sum_{i<j,j\neq\bar{i}} + \bar{g}_n(u)\sum_{i>j,j\neq\bar{i}}\right) E_{ij}\otimes E_{ji}\,, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
a_n(u) &= 2\sinh\left(\frac{u}{2}-2\eta\right)\cosh\left(\frac{u}{2}-(2n+1)\eta\right),\\
b_n(u) &= 2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2}-(2n+1)\eta\right),\\
c_n(u,i) &= 2e^{-u+2i\eta}\sinh((2i-(2n+1))\eta)\sinh(2\eta) - 2e^{(2i-(2n+1))\eta}\sinh(2\eta)\cosh(2i\eta)\,,\\
\bar{c}_n(u,i) &= 2e^{u-2\bar{i}\eta}\sinh(((2n+1)-2\bar{i})\eta)\sinh(2\eta) - 2e^{((2n+1)-2\bar{i})\eta}\sinh(2\eta)\cosh(2\bar{i}\eta),\\
d_n(u,i,j) &= -2e^{-\frac{u}{2}+(2n+1+2(\tilde{i}-\tilde{j}))\eta}\sinh(2\eta)\sinh\left(\frac{u}{2}\right),\\
\bar{d}_n(u,i,j) &= 2e^{\frac{u}{2}+(2(\tilde{i}-\tilde{j})-2n-1)\eta}\sinh(2\eta)\sinh\left(\frac{u}{2}\right),\\
e_n(u) &= 2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2}-(2n-1)\eta\right),\\
f_n(u) &= b_n(u) - 2\sinh(2\eta)\cosh((2n+1)\eta),\\
g_n(u) &= -2e^{-\frac{u}{2}}\sinh(2\eta)\cosh\left(\frac{u}{2}-(2n+1)\eta\right),\\
\bar{g}_n(u) &= -2e^{\frac{u}{2}}\sinh(2\eta)\cosh\left(\frac{u}{2}-(2n+1)\eta\right).
\end{aligned} \tag{2.2}$$

$$i+\bar{i} = 2n+2\,,\qquad \tilde{i} = \begin{cases} i+\dfrac{1}{2}, & 1\le i<n+1\\ i, & i=n+1\\ i-\dfrac{1}{2}, & n+1<i\le 2n+1\end{cases} \tag{2.3}$$

This R matrix satisfies the following properties

$$
\begin{aligned}
&\text{regularity}: R_{12}^{(n)}(0) = \rho_n(0)^{1/2}\mathcal{P}_{12}\,,\\
&\text{unitarity}: R_{12}^{(n)}(u)R_{21}^{(n)}(-u) = \rho_n(u)\,,\\
&\text{PT-symmetry}: \mathcal{P}_{12}R_{12}^{(n)}(u)\mathcal{P}_{12} = \left[R_{12}^{(n)}\right]^{t_1 t_2}(u)\,,\\
&\text{crossing-unitarity}: M_1 R_{12}^{(n)}(u)^{t_2} M_1^{-1} R_{12}^{(n)}(-u-2\xi_n)^{t_1} = \rho_n(u+\xi_n)\,,
\end{aligned}
$$

with $\rho_n(u) = a_n(u)a_n(-u)$, $\xi_n = -\sqrt{-1}\pi - 2(2n+1)\eta$, $M^i_j = \delta_{ij}e^{4(n+1-\tilde{i})\eta}$. The exchange operator $\mathcal{P}$ is given by $\mathcal{P}^{ij}_{kl} = \delta_{il}\delta_{jk}$, and t_i denotes the transposition in i-th space. It also satisfies the Yang-Baxter equation(YBE) [6]

$$
R_{12}^{(n)}(u-v)R_{13}^{(n)}(u)R_{23}^{(n)}(v) = R_{23}^{(n)}(v)R_{13}^{(n)}(u)R_{12}^{(n)}(u-v)\,, \tag{2.4}
$$

$R_{12}^{(n)}(u) = R^{(n)}(u)\otimes 1$, $R_{23}^{(n)}(u) = 1\otimes R^{(n)}(u)$ etc. $R_{21}^{(n)} = \mathcal{P}_{12}R_{12}^{(n)}\mathcal{P}_{12}$. Define the following reflection equations

$$
R_{12}^{(n)}(u-v)\overset{1}{K}_-(u)R_{21}^{(n)}(u+v)\overset{2}{K}_-(v) = \overset{2}{K}_-(v)R_{12}^{(n)}(u+v)\overset{1}{K}_-(u)R_{21}^{(n)}(u-v)\,, \tag{2.5}
$$

$$
\begin{aligned}
&R_{12}^{(n)}(-u+v)\overset{1}{K}{}_+^{t_1}(u)\,\overset{1}{M}{}^{-1}\,R_{21}^{(n)}(-u-v-2\xi_n)\,\overset{1}{M}\overset{2}{K}{}_+^{t_2}(v) =\\
&\qquad = \overset{2}{K}{}_+^{t_2}(v)\,\overset{1}{M}\,R_{12}^{(n)}(-u-v-2\xi_n)\,\overset{1}{M}{}^{-1}\overset{1}{K}{}_+^{t_1}(u)R_{21}^{(n)}(-u+v)\,,
\end{aligned} \tag{2.6}
$$

where $\overset{1}{K}_\pm(u) = K_\pm(u)\otimes 1$, $\overset{2}{K}_\pm(u) = 1\otimes K_\pm(u)$. Then the transfer matrix defined as

$$
t(u) = \mathrm{tr}K_+(u)U(u) \tag{2.7}
$$

constitutes an one-parameter commutative family, i.e. $[t(u), t(v)] = 0$. Here

$$
U(u) = T(u)K_-(u)T^{-1}(-u)\,, \tag{2.8}
$$

$$
T(u) = R_{01}^{(n)}(u)R_{02}^{(n)}\cdots R_{0N}^{(n)}(u)\,. \tag{2.9}
$$

The corresponding integrable open chain hamiltonian takes the form

$$
H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2}\overset{1}{K'}_-(0) + \frac{\mathrm{tr}\,\overset{0}{K}_+(0)H_{N,0}}{\mathrm{tr}K_+(0)}\,, \tag{2.10}
$$

with $H_{k,k+1} = \mathcal{P}_{k,k+1}R'_{kk+1}(u)|_{u=0}$. The general solutions of eq. (2.5) have been obtained in ref. [23]. The trivial and nontrivial diagonal reflecting matrices take the form

$$
K_-^{(1)}(u,n)_i = 1\,, \qquad i = 1,2,\ldots,2n+1 \tag{2.11}
$$

$$
K_-^{(2)}(u,n,p_-)_i = \begin{cases} e^{-u}\left[c_-\cosh(\eta) + \sinh(u-2(2p_- - n)\eta)\right], & (1\le i\le p_-)\\ c_-\cosh(u+\eta) - \sinh(2(2p_- - n)\eta), & (p_- + 1 \le i \le 2n+1-p_-)\\ e^{u}\left[c_-\cosh(\eta) + \sinh(u-2(2p_- - n)\eta)\right]. & (2n-p_-+2\le i\le 2n+1)\end{cases} \tag{2.12}
$$

$$
K_+^{(1)}(u,n)_i = e^{4(n+1-\tilde{i})\eta}\,, \qquad i = 1,2,\ldots,2n+1 \tag{2.13}
$$

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$$K_+^{(2)}(u,n,p_+)_i = \begin{cases} e^{4(n+1-\tilde{i})\eta+u-2(2n+1)\eta}\left[c_+\cosh(\eta)+\sinh(u-2(3n-2p_++1)\eta)\right], \\ \qquad (1\le i\le p_+) \\ e^{4(n+1-\tilde{i})\eta}\left[c_+\cosh(u-(4n+3)\eta)+\sinh(2(2p_+-n)\eta)\right], \\ \qquad (p_++1\le i\le 2n+1-p_+) \\ e^{4(n+1-\tilde{i})\eta-u+2(2n+1)\eta}\left[c_+\cosh(\eta)+\sinh(u-2(3n-2p_++1)\eta)\right], \\ \qquad (2n-p_++2\le i\le 2n+1), \end{cases} \tag{2.14}$$

where $c_-^2=c_+^2=-1$, $p_\pm$ are integer numbers varying from 1 to n.

3. The algebraical Bethe ansatz

In this section, we will present the main procedure of solving open $A_{2n}^{(2)}$ model by using the nested Bethe ansatz. We begin with introducing the vacuum state.

3.1 The vacuum state

Firstly, we write the double-monodromy matrix (2.8) as

$$U(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) & \cdots & B_{2n-1} & F(u) \\ D_1(u) & A_{11}(u) & A_{12}(u) & \cdots & A_{12n-1}(u) & E_1(u) \\ D_2(u) & A_{21}(u) & A_{22}(u) & \cdots & A_{22n-1}(u) & E_2(u) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{2n-1}(u) & A_{2n-11}(u) & A_{2n-12}(u) & \cdots & A_{2n-12n-1}(u) & E_{2n-1}(u) \\ G(u) & C_1(u) & C_2(u) & \cdots & C_{2n-1}(u) & A_2(u) \end{pmatrix}. \tag{3.1}$$

With the help of eqs. (2.4), (2.5), we can prove that $U(u)$ in eq. (3.1) satisfy the following equation

$$R_{12}^{(n)}(u-v)\overset{1}{U}(u)R_{21}^{(n)}(u+v)\overset{2}{U}(v) = \overset{2}{U}(v)R_{12}^{(n)}(u+v)\overset{1}{U}(u)R_{21}^{(n)}(u-v)\,. \tag{3.2}$$

Applying the double-row monodromy matrix eq. (3.1) on the vacuum state $|0\rangle=\prod^{\otimes N}(1,$ $0,\ldots,0)^t$, we can find

$$\begin{aligned} D_a(u)|0\rangle &= 0\,, & C_a(u)|0\rangle &= 0\,, & G(u)|0\rangle &= 0\,, \\ B_a(u)|0\rangle &\neq 0\,, & E_a(u)|0\rangle &\neq 0\,, & F(u)|0\rangle &\neq 0\,, \\ A_{aa}(u)|0\rangle &\neq 0\,, & A_{ab}(u)|0\rangle &= 0 & &(a\neq b)\,, \\ A(u)|0\rangle &\neq 0\,, & A_2(u)|0\rangle &\neq 0 & &(a=1,2,\ldots,2n-1)\,. \end{aligned} \tag{3.3}$$

From eq. (3.3), we can see that D_a, C_a, G and B_a, E_a, F play the role of annihilation operators and creation operators on the vacuum state, respectively. The A, A_{aa}, A_2 are diagonal operators on the vacuum state. Introducing two new operators (a not very long calculation is omitted here)

$$\tilde{A}_{ab}(u) = A_{ab}(u)-\tilde{f}_1(u)A(u)\delta_{ab}\,, \tag{3.4}$$

$$\tilde{A}_2(u) = A_2(u)-\tilde{f}_3(u)A(u)-\tilde{f}_2(u)\sum_{a=1}^{2n-1}e^{4(n-\tilde{a})\eta}\tilde{A}_{aa}(u)\,, \tag{3.5}$$

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where

$$\tilde{f}_1(u) = \frac{\bar{g}_n(2u)}{a_n(2u)}\,, \qquad \tilde{f}_3(u) = \frac{\bar{c}_n(2u, 2n+1)}{a_n(2u)}\,, \qquad \tilde{f}_2(u) = -\frac{e^{u-4\eta}\sinh(2\eta)}{\sinh(u-4n\eta)}\,,$$

$$\tilde{a} = \begin{cases} a+\frac{1}{2}\,, & 1 \leq a < n \\ a\,, & a = n \\ a-\frac{1}{2}\,, & n+1 < a \leq 2n-1 \end{cases} \tag{3.6}$$

we have

$$A(u)|0\rangle = K_-(u)_1[a_n(u)]^{2N}\rho_n(u)^{-N}|0\rangle = \omega_1(u)|0\rangle\,, \tag{3.7}$$

$$\tilde{A}_{aa}(u)|0\rangle = (K_-(u)_{a+1} - \tilde{f}_1(u)K_-(u)_1)[b_n(u)]^{2N}\rho_n(u)^{-N}|0\rangle = k^-(u)_a\omega(u)|0\rangle\,, \tag{3.8}$$

$$\begin{aligned}\tilde{A}_2(u)|0\rangle = \Big\{&K_-(u)_{2n+1} - \tilde{f}_2(u)\sum_{a=1}^{2n-1} e^{4(n-\tilde{a})\eta}\Big(K_-(u)_{a+1} - \tilde{f}_1(u)K_-(u)_1\Big) - \\ &- \tilde{f}_3(u)K_-(u)_1\Big\}[e_n](u)^{2N}\rho_n(u)^{-N}|0\rangle = \omega_{2n+1}(u)|0\rangle\,.\end{aligned} \tag{3.9}$$

In terms of new operators, the transfer matrix (2.7) can be rewritten as

$$t(u) = w_1(u)A(u) + \sum_{a=1}^{2n-1} w(u)k_a^+(u)\tilde{A}_{aa}(u) + w_{2n+1}(u)\tilde{A}_2(u) \tag{3.10}$$

with

$$\begin{aligned} w_1(u) &= K_+(u)_1 + \tilde{f}_3(u)K_+(u)_{2n+1} + \tilde{f}_1(u)\sum_{a=1}^{2n-1}K_+(u)_{a+1}\,, \\ w(u)k_a^+(u) &= K_+(u)_{a+1} + e^{4(n-\tilde{a})\eta}\tilde{f}_2(u)K_+(u)_{2n+1}\,, \qquad w_{2n+1}(u) = K_+(u)_{2n+1}\,. \end{aligned} \tag{3.11}$$

The explicit expression of coefficient functions ω's and w's can be seen at the case $j = 0$ in appendix B, $k^{\mp}(u) = K^{(1)}_{\mp}(\tilde{u}, n-1)$ or $K^{(2)}_{\mp}(\tilde{u}, n-1, p_{\mp}-1)$ depend on the choice of boundary, $\tilde{u} = u - 2\eta$.

3.2 The Fundamental commutation relations

In order to construct the general m-particle state, we need to find the commutation relations between the creation, diagonal and annihilation fields. Here we only provide some important commutation relations. Taking some components of eq. (3.2), we can obtain the following fundamental commutation relations

$$\begin{aligned} B_a(u)B_b(v) \;+\; &\delta_{\bar{a}b}g_1(u,v,a)F(u)A(v) + g_2(u,v,a)F(u)\tilde{A}_{\bar{a}b}(v) = \\ &= \hat{r}(u_-)^{dc}_{ba}[B_d(v)B_c(u) + \delta_{\bar{d}c}g_1(v,u,d)F(v)A(u) + g_2(v,u,d)F(v)\tilde{A}_{\bar{d}c}(u)]\,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} A(u)B_a(v) &= a_1^1(u,v)B_a(v)A(u) + a_2^1(u,v)B_a(u)A(v) + a_3^1(u,v)B_d(u)\tilde{A}_{da}(v) + \\ &\quad + a_4^1(u,v,\bar{a})F(u)D_{\bar{a}}(v) + a_5^1(u,v)F(u)C_a(v) + a_6^1(u,v,\bar{a})F(v)D_{\bar{a}}(u)\,, \end{aligned} \tag{3.13}$$

$$\tilde{A}_{ab}(u)B_c(v) = \tilde{r}(u_+)^{ae}_{dg}\bar{r}(u_-)^{gf}_{cb}B_e(v)\tilde{A}_{df}(u) + R_1^A(u,v)^{af}_{cb}B_f(u)A(v) +$$

$$
\begin{aligned}
& + R_2^A(u,v)_{db}^{af} B_f(u)\tilde{A}_{dc}(v) + \delta_{\bar{b}c} R_3^A(u,v,\bar{b}) E_a(u) A(v) + \\
& + R_4^A(u,v,\bar{b}) E_a(u)\tilde{A}_{\bar{b}c}(v) + R_5^A(u,v)_{cb}^{af} F(u) D_{\bar{f}}(v) + \\
& + \delta_{ab} R_6^A(u,v) F(u) C_c(v) + R_7^A(u,v)_{cb}^{af} F(v) D_{\bar{f}}(u) + \\
& + R_8^A(u,v)_{cb}^{af} F(v) C_f(u)\,, \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}_2(u) B_a(v) = {} & a_1^3(u,v) B_a(v)\tilde{A}_2(u) + a_2^3(u,v) B_a(u) A(v) + a_3^3(u,v) B_d(u)\tilde{A}_{da}(v) + \\
& + a_4^3(u,v,a) E_{\bar{a}}(u) A(v) + a_5^3(u,v,\bar{d}) E_d(u)\tilde{A}_{\bar{d}a}(v) + a_6^3(u,v,\bar{a}) F(u) D_{\bar{a}}(v) + \\
& + a_7^3(u,v) F(u) C_a(v) + a_8^3(u,v,\bar{a}) F(v) D_{\bar{a}}(u) + a_9^3(u,v) F(v) C_a(u)\,, \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
A(u) F(v) = {} & b_1^1(u,v) F(v) A(u) + b_2^1(u,v) F(u) A(v) + b_3^1(u,v,d) F(u)\tilde{A}_{dd}(v) + \\
& + b_4^1(u,v) F(u)\tilde{A}_2(v) + b_5^1(u,v,d) B_{\bar{d}}(u) B_d(v) + b_6^1(u,v) B_d(u) E_d(v)\,, \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}_{ab}(u) F(v) = {} & b_1^2(u,v) F(v)\tilde{A}_{ab}(u) + \delta_{ab} b_2^2(u,v) F(u) A(v) + R_1^F(u,v)_{b\bar{a}}^{dc} F(u)\tilde{A}_{\bar{d}c}(v) + \\
& + \delta_{ab} b_3^2(u,v) F(u)\tilde{A}_2(v) + R_2^F(u,v)_{b\bar{a}}^{dc} B_d(u) B_c(v) + R_3^F(u,v)_{db}^{ac} B_c(u) E_d(v) + \\
& + b_4^2(u,v) E_a(u) B_b(v) + b_5^2(u,v,\bar{b}) E_a(u) E_{\bar{b}}(v)\,, \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}_2(u) F(v) = {} & b_1^3(u,v) F(v)\tilde{A}_2(u) + b_2^3(u,v) F(u) A(v) + b_3^3(u,v,d) F(u)\tilde{A}_{dd}(v) + \\
& + b_4^3(u,v) F(u)\tilde{A}_2(v) + b_5^3(u,v,\bar{d}) B_{\bar{d}}(u) B_d(v) + b_6^3(u,v) B_d(u) E_d(v) + \\
& + b_7^3(u,v) E_d(u) B_d(v) + b_8^3(u,v,d) E_d(u) E_{\bar{d}}(v)\,. \qquad (3.18)
\end{aligned}
$$

Besides the above fundamental commutation relations, we also need the following necessary commutation relations

$$
\begin{aligned}
D_a(u) B_b(v) = {} & R_1^D(u,v)_{db}^{ac} B_c(v) D_d(u) + c_1^1(u,v,\bar{a}) B_{\bar{a}}(v) C_b(u) + \delta_{ab} c_2^1(u,v) F(v) G(u) + \\
& + c_3^1(u,v,\bar{a}) B_{\bar{a}}(u) C_b(v) + c_4^1(u,v) E_a(u) C_b(v) + \delta_{ab} c_5^1(u,v) A(v) A(u) + \\
& + c_6^1(u,v) A(v)\tilde{A}_{ab}(u) + \delta_{ab} c_7^1(u,v) A(u) A(v) + c_8^1(u,v) A(u)\tilde{A}_{ab}(v) + \\
& + c_9^1(u,v)\tilde{A}_{ab}(u) A(v) + c_{10}^1(u,v)\tilde{A}_{ad}(u)\tilde{A}_{db}(v)\,, \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
C_a(u) B_b(v) = {} & R_1^C(u,v)_{ba}^{dc} B_d(v) C_c(u) + R_2^C(u,v)_{db}^{\bar{a}c} B_c(v) D_d(u) + \delta_{a\bar{b}} c_1^2(u,v,\bar{a}) F(v) G(u) + \\
& + c_2^2(u,v) B_a(u) C_b(v) + c_3^2(u,v,\bar{a}) E_{\bar{a}}(u) C_b(v) + \delta_{a\bar{b}} c_4^2(u,v,\bar{a}) A(v) A(u) + \\
& + R_3^C(u,v)_{ba}^{dc} A(v)\tilde{A}_{\bar{d}c}(u) + \delta_{a\bar{b}} c_5^2(u,v,\bar{a}) A(v)\tilde{A}_2(u) + \\
& + \delta_{a\bar{b}} c_6^2(u,v,\bar{a}) A(u) A(v) + c_7^2(u,v,\bar{a}) A(u)\tilde{A}_{\bar{a}b}(v) + \\
& + R_4^C(u,v)_{ba}^{dc}\tilde{A}_{\bar{d}c}(u) A(v) + R_5^C(u,v)_{ea}^{dc}\tilde{A}_{\bar{d}c}(u)\tilde{A}_{eb}(v) + \\
& + \delta_{a\bar{b}} c_8^2(u,v,\bar{a})\tilde{A}_2(u) A(v) + c_9^2(u,v,\bar{a})\tilde{A}_2(u)\tilde{A}_{\bar{a}b}(v)\,, \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
B_a(u) E_b(v) = {} & R_1^{be}(u,v)_{bd}^{ca} E_c(v) B_d(u) + R_2^{be}(u,v)_{\bar{b}a}^{dc} B_d(v) B_c(u) + \delta_{ab} e_1^1(u,v,a) F(v) A(u) + \\
& + R_3^{be}(u,v)_{\bar{b}a}^{dc} F(v)\tilde{A}_{\bar{d}c}(u) + \delta_{ab} e_2^1(u,v,a) F(u) A(v) + \\
& + R_4^{be}(u,v)_{bd}^{ca} F(u)\tilde{A}_{cd}(v) + \delta_{ab} e_3^1(u,v,a) F(u)\tilde{A}_2(v)\,, \qquad (3.21)
\end{aligned}
$$

where all the repeated indices sum over 1 to $2n-1$, $u_\pm = u \pm v$ and

$$
g_1(u,v,a) = -\frac{d_n(u_-,1,\bar{a}) b_n(2v)}{e_n(u_-) a_n(2v)}\,, \qquad g_2(u,v,a) = \frac{d_n(u_+,1,\bar{a})}{b_n(u_+)} \qquad a + \bar{a} = 2n\,. \quad (3.22)
$$

The $\hat{r}(u), \tilde{r}(u)$ and $\bar{r}(u)$ are given by

$$\hat{r}(u) = \frac{1}{e_n(u)} \frac{b_n(u)}{a_n(u)} R^{(n-1)}(u), \quad \tilde{r}(u) = \frac{1}{a_n(u)} R^{(n-1)}(u - 4\eta), \quad \bar{r}(u) = \frac{1}{e_n(u)} R^{(n-1)}(u),$$

respectively. The other coefficients such as $a_1^1(u,v), R_1^A(u,v)_{cb}^{af}$ etc. are not presented here for their long and tedious expressions.

3.3 The m-particle state

Inferred from the commutation relation eq. (3.12), we can construct the general m-particle state as follow. Let

$$\begin{aligned}
\Phi_m^{b_1\cdots b_m}(v_1,\ldots,v_m) = {}& B_{b_1}(v_1)\Phi_{m-1}^{b_2\cdots b_m}(v_2,\ldots,v_m) + \\
& + F(v_1)\sum_{i=2}^{m} \Phi_{m-2}^{d_3\cdots d_m}(v_2,\ldots,\check{v}_i,\ldots,v_m) S_{b_2\cdots b_m}^{d_2\cdots d_m}(v_i;\{\check{v}_1,\check{v}_i\}) \times \\
& \times \Lambda_1^{m-2}(v_i;\{\check{v}_1,\check{v}_i\}) g_1(v_1,v_i,b_1) A(v_i)\delta_{\bar{b}_1 d_2} + \\
& + F(v_1)\sum_{i=2}^{m} \Phi_{m-2}^{d_3\cdot d_m}(v_2,\ldots,\check{v}_i,\ldots,v_m)[\tilde{T}^{m-2}(v_i;\{\check{v}_1,\check{v}_i\})_{c_3\cdots c_m}^{d_3\cdot d_m}]_{\bar{b}_1 c_2} \times \\
& \times S_{b_2\cdots b_m}^{c_2\cdots c_m}(v_i;\{\check{v}_1,\check{v}_i\}) g_2(v_1,v_i,b_1), \qquad (3.23)
\end{aligned}$$

where

$$\begin{aligned}
S_{b_1\cdots b_m}^{d_1\cdots d_m}(v_i;\{\check{v}_i\}) &= \hat{r}_{c_2 b_1}^{d_1 d_2}(v_1 - v_i)\hat{r}_{c_3 b_2}^{c_2 d_3}(v_2 - v_i)\cdots \hat{r}_{b_i b_{i-1}}^{c_{i-1} d_i}(v_{i-1} - v_i) \prod_{j=i+1}^{m} \delta_{d_j b_j} \\
[\tilde{T}^m(u;\{v_m\})_{c_1\cdots c_m}^{d_1\cdots d_m}]_{ab} &= \tilde{r}_{h_1 g_1}^{a d_1}(u+v_1)\tilde{r}_{h_2 g_2}^{h_1 d_2}(u+v_2)\cdots \tilde{r}_{h_m g_m}^{h_{m-1} d_m}(u+v_m)\tilde{A}_{h_m f_m}(u) \times \\
& \quad \times \bar{r}_{c_m f_{m-1}}^{g_m f_m}(u - v_m)\bar{r}_{c_{m-1} f_{m-2}}^{g_{m-1} f_{m-1}}(u - v_{m-1})\cdots \bar{r}_{c_1 b}^{g_1 f_1}(u - v_1) \qquad (3.24)
\end{aligned}$$

with $S_{b_1\cdots b_m}^{d_1\cdots d_m}(v_1; v_2,\ldots,v_m) = \prod_{i=1}^m \delta_{d_i b_i}$, $[\tilde{T}^0(u)]_{ab} = \tilde{A}_{ab}(u)$, $\Lambda_l^m(u; v_1, v_2,\ldots,v_m) = \prod_{i=1}^m a_1^l(u,v_i)$, $(l = 1,3)$, $\Phi_0 = 1, \Phi_1^{b_1}(v_1) = B_{b_1}(v_1)$. The $\check{v}_i$ means missing of v_i in the sequence $v_1, v_2,\ldots,v_m$.

Then the general m-particle state is defined by

$$|\Upsilon_m(v_1,\ldots,v_m)\rangle = \Phi_m^{b_1\cdots b_m}(v_1,\ldots,v_m)F^{b_1\cdots b_m}|0\rangle, \qquad (3.25)$$

which enjoys the property

$$\begin{aligned}
&\Phi_m^{b_1\cdots b_i b_{i+1}\cdots b_m}(v_1,\ldots,v_i,v_{i+1},\ldots,v_m)F^{b_1\cdots b_m}|0\rangle = \\
&\quad = \Phi_m^{b_1\cdots a_i a_{i+1}\cdots b_m}(v_1,\ldots,v_{i+1},v_i,\ldots,v_m)\hat{r}_{b_i b_{i+1}}^{a_{i+1} a_i}(v_i - v_{i+1})F^{b_1\cdots b_m}|0\rangle. \qquad (3.26)
\end{aligned}$$

It is easy to verify eq. (3.26) excepting $i = 1$. But, the proof for the case $i = 1$ becomes very involved and are omitted here.

3.4 The eigenvalue and Bethe equations

We can apply the operators A 's on the eigenstate ansatz and obtain (see appendix A)

$$\begin{aligned}
x(u)|\Upsilon_m(v_1,\ldots,v_m)\rangle &= |\tilde{\Psi}_x(u,\{v_m\})\rangle + \\
&+\sum_{i=1}^{m} h_1^x(u,v_i,d_1)|\Psi_{m-1}^{(1)}(u,v_i;\{v_m\})_{d_1d_1}\rangle + \\
&+\sum_{i=1}^{m} h_2^x(u,v_i,d)|\tilde{\Psi}_{m-1}^{(2)}(u,v_i;\{v_m\})_{dd}\rangle + \\
&+\sum_{i=1}^{m} h_3^x(u,v_i,\bar{\alpha}_x)|\Psi_{m-1}^{(3)}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle + \\
&+\sum_{i=1}^{m} h_4^x(u,v_i,\bar{\alpha}_x)|\tilde{\Psi}_{m-1}^{(4)}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle + \\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \tilde{H}_{1,d_1}^x(u,v_i,v_j)|\tilde{\Psi}_{m-2}^{(5)}(u,v_i,v_j;\{v_m\})_{d_1}\rangle + \\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \tilde{H}_{2,d_1}^x(u,v_i,v_j)|\tilde{\Psi}_{m-2}^{(6)}(u,v_i,v_j;\{v_m\})_{d_1}\rangle + \\
&+\sum_{i=1}^{n-1}\sum_{j=i+1}^{m} \tilde{H}_{3,d_1}^x(u,v_i,v_j)|\tilde{\Psi}_{m-2}^{(7)}(u,v_i,v_j;\{v_m\})_{d_1}\rangle + \\
&+\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \tilde{H}_{4,d_1}^x(u,v_i,v_j)|\tilde{\Psi}_{m-2}^{(8)}(u,v_i,v_j;\{v_m\})_{d_1}\rangle\,, \qquad (3.27)
\end{aligned}$$

where the expression of $|\tilde{\Psi}\rangle$'s and coefficients $\tilde{H}_{j,d_1}^x$ $(j=1,2,3,4)$ are given in appendix A, $x=A,\tilde{A}_{aa},\tilde{A}_2$. Using eq.(3.10), we then get

$$\begin{aligned}
t(u)|\Upsilon_m(v_1,\ldots,v_m)\rangle &= w_1(u)\omega_1(u)\Lambda_1^m(u;v_1,\ldots,v_m)|\Upsilon_m(v_1,\ldots,v_m)\rangle + \\
&+w(u)\omega(u)\Lambda_2^m(u;v_1,\ldots,v_m)\Phi_m^{d_1\cdots d_m}(v_1,\ldots,v_m)\times \\
&\times\tau_1(\tilde{u};\{\tilde{v}_m\})_{b_1\cdots b_m}^{d_1\cdots d_m}F^{b_1\cdots b_m}|0\rangle + \qquad (3.28)\\
&+w_{2n+1}(u)\omega_{2n+1}(u)\Lambda_3^m(u;v_1,\ldots,v_m)|\Upsilon_m(v_1,\ldots,v_m)\rangle + \text{u.t.}\,,
\end{aligned}$$

where u.t. denotes the unwanted terms,

$$\begin{aligned}
\tau_1(\tilde{u};\{\tilde{v}_m\})_{c_1\cdots c_m}^{d_1\cdots d_m} &= k^+(u)_a L(\tilde{u},\tilde{v}_1)_{h_1g_1}^{ad_1}L(\tilde{u},\tilde{v}_2)_{h_2g_2}^{h_1d_2}\cdots L(\tilde{u},\tilde{v}_m)_{h_mg_m}^{h_{m-1}d_m}k^-(u)_{h_m}\times \qquad (3.29)\\
&\times L^{-1}(-\tilde{u},\tilde{v}_m)_{f_{m-1}c_m}^{h_mg_m}L^{-1}(-\tilde{u},\tilde{v}_{m-1})_{f_{m-2}c_{m-1}}^{f_{m-1}g_{m-1}}\cdots L^{-1}(-\tilde{u},\tilde{v}_1)_{ac_1}^{f_1g_1}\,,
\end{aligned}$$

with $\tilde{v}_i = v_i - 2\eta$, and

$$\begin{aligned}
L(\tilde{u},\tilde{v})_{cd}^{ab} &= R^{(n-1)}(\tilde{u}+\tilde{v})_{cd}^{ab}\,,\\
L^{-1}(-\tilde{u},\tilde{v})_{cd}^{ab} &= \frac{R^{(n-1)}(\tilde{u}-\tilde{v})_{dc}^{ba}}{\rho_{n-1}(\tilde{u}-\tilde{v})}\,. \qquad (3.30)
\end{aligned}$$

Thus, we get the conclusion that$|\Upsilon_m(v_1,\ldots,v_m)\rangle$ is the eigenstate of $t(u)$, i.e.

$$\begin{aligned}
t(u)|\Upsilon_m(v_1,\ldots,v_m)\rangle &= \{w_1(u)\omega_1(u)\Lambda_1^m(u;v_1,\ldots,v_m)+ \\
&\quad + w(u)\omega(u)\Lambda_2^m(u;v_1,\ldots,v_m)\Gamma_1(\tilde{u};\{\tilde{v}_m\};\{v_{m_1}^{(1)}\})+ \\
&\quad + w_{2n+1}(u)\omega_{2n+1}(u)\Lambda_3^m(u;v_1,\ldots,v_m)\}|\Upsilon_m(v_1,\ldots,v_m)\rangle \\
&= \Gamma(u;\{v_m\})|\Upsilon_m(v_1,\ldots,v_m)\rangle\,, \qquad (3.31)
\end{aligned}$$

if the parameters satisfy

$$\tau_1(\tilde{u};\{\tilde{v}_m\})F^{b_1\cdots b_m} = \Gamma_1(\tilde{u};\{\tilde{v}_m\};\{v_{m_1}^{(1)}\})F^{b_1\cdots b_m}\,, \qquad (3.32)$$

$$\Gamma_1(\tilde{v}_i;\{\tilde{v}_m\};\{v_{m_1}^{(1)}\}) = -\rho_{n-1}^{-1/2}(0)\frac{\omega_1(v_i)\Lambda_1^{m-1}(v_i;\{\check{v}_i\})}{\omega(v_i)\Lambda_2^{m-1}(v_i;\{\check{v}_i\})}\beta_1(v_i),\quad (i=1,\ldots,m)\,, \qquad (3.33)$$

where

$$\beta_1(v_i)=\begin{cases} -\dfrac{2e^{2\eta}\sinh(v_i)\sinh(v_i-4n\eta)\cosh(v_i-(2n-1)\eta)}{\sinh(v_i-2\eta)}, & \text{for the eq.(2.13)} \\ -\dfrac{2e^{v_i}\sinh(v_i)\sinh(v_i-4n\eta)}{\sinh(v_i-2\eta)}[\sinh(2\eta)+c_+\cosh(v_i-(2n+1)\eta)]\times & \\ \quad\times[\cosh((4p_+-4n-1)\eta)-c_+\sinh(v_i-2(n+1)\eta)]. & \text{for the eq.(2.14)} \end{cases}. \qquad (3.34)$$

All unwanted terms cancel out by the following three kinds of identities

$$\begin{aligned}
&\beta_1(v_i)=T^{(d_1)}(v_i)\frac{w_1(u)a_2^1(u,v_i)+\sum_{d=1}^{2n-1}w_{d+1}(u)R_1^A(u,v_i)_{d_1d}^{dd_1}+w_{2n+1}(u)a_2^3(u,v_i)}{w_1(u)a_3^1(u,v_i)+\sum_{d=1}^{2n-1}w_{d+1}(u)R_2^A(u,v_i)_{d_1d}^{dd_1}+w_{2n+1}(u)a_3^3(u,v_i)}\,,\\
&\beta_1(v_i)=T^{(d_1)}(v_i)\frac{w_{\bar{d}_1+1}(u)R_3^A(u,v_i,d_1)+w_{2n+1}(u)a_4^3(u,v_i,d_1)}{w_{\bar{d}_1+1}(u)R_4^A(u,v_i,d_1)+w_{2n+1}(u)a_5^3(u,v_i,d_1)}\,,\\
&\sum_{l=1}^{2n+1}w_l(u)\tilde{H}_{1,d_1}^{x_l}(u,v_i,v_j)-\left[\sum_{l=1}^{2n+1}w_l(u)\tilde{H}_{2,d_1}^{x_l}(u,v_i,v_j)\right]\beta_1(v_i)- \qquad (3.35)\\
&\quad-\left[\sum_{l=1}^{2n+1}w_l(u)\tilde{H}_{3,d_1}^{x_l}(u,v_i,v_j)\right]\beta_1(v_j)+\left[\sum_{l=1}^{2n+1}w_l(u)\tilde{H}_{4,d_1}^{x_l}(u,v_i,v_j)\right]\beta_1(v_i)\beta_1(v_j)=0
\end{aligned}$$

with $x_1 = A, x_{l+1} = \tilde{A}_{ll}, x_{2n+1} = \tilde{A}_2,\ d_1 = 1,2,\ldots,2n-1$. From eqs. (3.29), (3.31) and (3.32) , we can see that the diagonalization of $\tau(u)$ is reduced to finding the eigenvalue of $\tau_1(\tilde{u};\{\tilde{v}_m\})$ which is just the transfer matrix of $A_{2(n-1)}^{(2)}$ vertex model with open boundary conditions. Repeating the procedure j times, we can obtain the $\Gamma_j(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\};\{v_{m_j}^{(j)}\})$ corresponding to the eigenvalue of open boundary $A_{2(n-j)}^{(2)}$ vertex model,

$$\begin{aligned}
\Gamma_j(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\};\{v_{m_j}^{(j)}\}) &= w_1^{(j)}(u^{(j)})\omega_1^{(j)}(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_1^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})+ \\
&\quad + w^{(j)}(u^{(j)})\omega^{(j)}(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_2^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})\times \\
&\quad \times\Gamma_{j+1}(u^{(j+1)};\{\tilde{v}_{m_j}^{(j)}\};\{v_{m_{j+1}}^{(j+1)}\})+ \qquad (3.36)\\
&\quad + w_{2(n-j)+1}^{(j)}(u^{(j)})\omega_{2(n-j)+1}^{(j)}(u^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_3^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})\,,
\end{aligned}$$

with $u^{(j)} = u - 2j\eta$, $\tilde{v}_k^{(j)} = v_k^{(j)} - 2\eta$, $\{v_{m_j}^{(j)}\} = \{v_1^{(j)}, \ldots, v_{m_j}^{(j)}\}$. $\{v_{m_0}^{(0)}\} = \{v_m\}$, $\{v_{m_{-1}}^{(-1)}\} = \{\tilde{v}_{m_{-1}}^{(-1)}\} = \{0\}$, $m_{-1} = N, m_0 = m$. Replacing the $m, u, \{v_m\}, n$ in the $\Lambda_l^m(u;\{v_m\})(l = 1,2,3)$ by $m_j, u^{(j)}, \{v_{m_j}^{(j)}\}, n-j$ respectively, we have $\Lambda_l^{m_j}(u^{(j)};\{v_{m_j}^{(j)}\})$. The Bethe equations are

$$\begin{aligned}
&\Gamma_{j+1}(\tilde{v}_i^{(j)};\{\tilde{v}_{m_j}^{(j)}\};\{v_{m_{j+1}}^{(j+1)}\}) = \qquad (3.37)\\
&= -\rho_{n-j-1}^{-\frac{1}{2}}(0)\frac{\omega_1^{(j)}(v_i^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_1^{m_j-1}(v_i^{(j)};\{\breve{v}_i^{(j)}\})}{\omega^{(j)}(v_i^{(j)};\{\tilde{v}_{m_{j-1}}^{(j-1)}\})\Lambda_2^{m_j-1}(v_i^{(j)};\{\breve{v}_i^{(j)}\})}\beta_{j+1}(v_i^{(j)}). \quad (i = 1,\ldots,m_j)\,.
\end{aligned}$$

The coefficients w's, ω's, β_j and the following $\bar{\omega}$, ξ's are expressed in appendix B. We can rewrite the eigenvalue in detail, which is

$$\begin{aligned}
\Gamma(u) &= \Gamma_0(u)\\
&= w_1^{(0)}\left(u^{(0)}\right)\bar{\omega}_1^{(0)}\left(u^{(0)}\right)\xi_1^{(0)}\left(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\}\right)\mathcal{A}^{(m_0)}(u) +\\
&\quad + w_{2n+1}^{(0)}\left(u^{(0)}\right)\bar{\omega}_{2n+1}^{(0)}\left(u^{(0)}\right)\xi_3^{(0)}\left(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\}\right)\mathcal{C}^{(m_0)}(u) +\\
&\quad + \sum_{j=0}^{n-2}\mu_j\left(u^{(j)}\right)\nu_j\left(u^{(j)}\right)w_1^{(j+1)}\left(u^{(j+1)}\right)\bar{\omega}_1^{(j+1)}\left(u^{(j+1)}\right)\times\\
&\quad \times\xi_2^{(0)}\left(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\}\right)\mathcal{B}^{(m_j,m_{j+1})}(u) +\\
&\quad + \sum_{j=0}^{n-2}\mu_j\left(u^{(j)}\right)\nu_j(u^{(j)})w_{2(n-j-1)+1}^{(j+1)}\left(u^{(j+1)}\right)\bar{\omega}_{2(n-j-1)+1}^{(j+1)}\left(u^{(j+1)}\right)\times\\
&\quad \times\xi_2^{(0)}\left(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\}\right)\bar{\mathcal{B}}^{(m_j,m_{j+1})}(u) +\\
&\quad + \mu_{n-1}\left(u^{(n-1)}\right)\nu_{n-1}\left(u^{(n-1)}\right)\xi_2^{(0)}\left(u^{(0)};\{\tilde{v}_{m_{-1}}^{(-1)}\}\right)\mathcal{B}^{(m_{n-1})}(u)\,, \qquad (3.38)
\end{aligned}$$

where

$$\begin{aligned}
\mu_j(u^{(j)}) &= \prod_{i=0}^{j}\bar{\omega}^{(i)}(u^{(i)})\,, \qquad \nu_j(u^{(j)}) = \prod_{i=0}^{j}w^{(i)}(u^{(i)})\,, \qquad (3.39)\\
\mathcal{A}^{(m_0)}(u) &= \prod_{k=1}^{m_0}\frac{\sinh(\frac{1}{2}(\tilde{u}^{(0)}+\tilde{v}_k^{(0)})+2\eta)\sinh(\frac{1}{2}(\tilde{u}^{(0)}-\tilde{v}_k^{(0)})+2\eta)}{\sinh(\frac{1}{2}(\tilde{u}^{(0)}+\tilde{v}_k^{(0)}))\sinh(\frac{1}{2}(\tilde{u}^{(0)}-\tilde{v}_k^{(0)}))}\,,\\
\mathcal{C}^{(m_0)}(u) &= \prod_{k=1}^{m_0}\frac{\cosh(\frac{1}{2}(\tilde{u}^{(0)}+\tilde{v}_k^{(0)})-(2n+1)\eta)\cosh(\frac{1}{2}(\tilde{u}^{(0)}-\tilde{v}_k^{(0)})-(2n+1)\eta)}{\cosh(\frac{1}{2}(\tilde{u}^{(0)}+\tilde{v}_k^{(0)})-(2n-1)\eta)\cosh(\frac{1}{2}(\tilde{u}^{(0)}-\tilde{v}_k^{(0)})-(2n-1)\eta)}\,,\\
\mathcal{B}^{(m_j,m_{j+1})}(u) &= \prod_{k=1}^{m_j}\frac{\sinh(\frac{1}{2}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})-2\eta)\sinh(\frac{1}{2}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})-2\eta)}{\sinh(\frac{1}{2}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)}))\sinh(\frac{1}{2}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)}))}\times\\
&\quad\times\prod_{l=1}^{m_{j+1}}\frac{\sinh(\frac{1}{2}(\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})+2\eta)\sinh(\frac{1}{2}(\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})+2\eta)}{\sinh(\frac{1}{2}(\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)}))\sinh(\frac{1}{2}(\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)}))}\,,\\
\bar{\mathcal{B}}^{(m_j,m_{j+1})}(u) &= \prod_{k=1}^{m_j}\frac{\cosh(\frac{1}{2}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})-(2n-2j-3)\eta)\cosh(\frac{1}{2}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})-(2n-2j-3)\eta)}{\cosh(\frac{1}{2}(\tilde{u}^{(j)}+\tilde{v}_k^{(j)})-(2n-2j-1)\eta)\cosh(\frac{1}{2}(\tilde{u}^{(j)}-\tilde{v}_k^{(j)})-(2n-2j-1)\eta)}\times\\
&\quad\times\prod_{l=1}^{m_{j+1}}\frac{\cosh(\frac{1}{2}(\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})-(2n-2j-1)\eta)\cosh(\frac{1}{2}(\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})-(2n-2j-1)\eta)}{\cosh(\frac{1}{2}(\tilde{u}^{(j+1)}+\tilde{v}_l^{(j+1)})-(2n-2j-3)\eta)\cosh(\frac{1}{2}(\tilde{u}^{(j+1)}-\tilde{v}_l^{(j+1)})-(2n-2j-3)\eta)}\,,\\
&\quad(j = 0,1,2,\ldots,n-2)
\end{aligned}$$

$$\mathcal{B}^{(m_{n-1})}(u) = \prod_{k=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(\tilde{u}^{(n-1)}+\tilde{v}_k^{(n-1)})-2\eta)\sinh(\frac{1}{2}(\tilde{u}^{(n-1)}-\tilde{v}_k^{(n-1)})-2\eta)}{\sinh(\frac{1}{2}(\tilde{u}^{(n-1)}+\tilde{v}_k^{(n-1)}))\sinh(\frac{1}{2}(\tilde{u}^{(n-1)}-\tilde{v}_k^{(n-1)}))} \times$$
$$\times \frac{\cosh(\frac{1}{2}(\tilde{u}^{(n-1)}+\tilde{v}_k^{(n-1)})+\eta)\cosh(\frac{1}{2}(\tilde{u}^{(n-1)}-\tilde{v}_k^{(n-1)})+\eta)}{\cosh(\frac{1}{2}(\tilde{u}^{(n-1)}+\tilde{v}_k^{(n-1)})-\eta)\cosh(\frac{1}{2}(\tilde{u}^{(n-1)}-\tilde{v}_k^{(n-1)})-\eta)}. \tag{3.40}$$

The explicit expression of Bethe equations eq. (3.37) is

$$\prod_{k=1}^{m_{j-1}} \frac{\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_k^{(j-1)})-\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}-\tilde{v}_k^{(j-1)})-\eta)}{\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_k^{(j-1)})+\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}-\tilde{v}_k^{(j-1)})+\eta)} \times$$
$$\times \prod_{l=1}^{m_{j+1}} \frac{\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_l^{(j+1)})-\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}-\tilde{v}_l^{(j+1)})-\eta)}{\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_l^{(j+1)})+\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}-\tilde{v}_l^{(j+1)})+\eta)} \times$$
$$\times \prod_{s=1\neq i}^{m_j} \frac{\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})+2\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})+2\eta)}{\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})-2\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(j)}+\tilde{v}_s^{(j)})-2\eta))} =$$
$$= W^{(j)}(\tilde{v}_i^{(j)})\Omega^{(j)}(\tilde{v}_i^{(j)}), \quad (i=1,\ldots,m_j; j=0,1,2,\ldots,n-2) \tag{3.41}$$
$$\prod_{k=1}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})-\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})-\eta)}{\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_k^{(n-2)})+\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}-\tilde{v}_k^{(n-2)})+\eta)} \times$$
$$\times \prod_{l=1}^{m_{n-1}} \frac{\cosh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})-\eta)\cosh(\frac{1}{2}(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})-\eta)}{\cosh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})+\eta)\cosh(\frac{1}{2}(\tilde{v}_i^{(n-1)}-\tilde{v}_l^{(n-1)})+\eta)} \times$$
$$\times \frac{\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})+2\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})+2\eta)}{\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})-2\eta)\sinh(\frac{1}{2}(\tilde{v}_i^{(n-1)}+\tilde{v}_l^{(n-1)})-2\eta))} =$$
$$= W^{(n-1)}(\tilde{v}_i^{(n-1)})\Omega^{(n-1)}(\tilde{v}_i^{(n-1)}) \quad (i=1,\ldots,m_{n-1}) \tag{3.42}$$

with

$$W^{(j)}(\tilde{v}_i^{(j)}) = \begin{cases} -\dfrac{w_1^{(j+1)}(\tilde{v}_i^{(j)})a_{n-j-1}(2\tilde{v}_i^{(j)}))}{\beta_{j+1}(v_i^{(j)})}, & j \neq n-1 \\ -\dfrac{1}{\beta_n(v_i^{(n-1)})}, & j = n-1 \end{cases} \tag{3.43}$$

$$\Omega^{(j)}(\tilde{v}_i^{(j)}) = \begin{cases} \dfrac{\bar{\omega}^{(j)}(v_i^{(j)})\bar{\omega}_1^{(j+1)}(\tilde{v}_i^{(j)})}{\bar{\omega}_1^{(j)}(v_i^{(j)})}, & j \neq n-1 \\ \dfrac{\bar{\omega}^{(j)}(v_i^{(n-1)})}{\bar{\omega}_1^{(n-1)}(v_i^{(n-1)})}. & j = n-1 \end{cases} \tag{3.44}$$

Up to now, we have gotten the whole eigenvalues and the Bethe equations of transfer matrix for the $A_{2n}^{(2)}$ vertex model with open boundary condition.

4. Conclusions

In the framework of algebraic Bethe ansatz, we solve the $A_{2n}^{(2)}$ vertex model with general diagonal reflecting matrices. When the model is $U_q(B_n)$ quantum invariant, we find that our conclusion agrees with that obtained by analytic Bethe ansatz method [3]. It seems that other models, such as $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $C_n^{(1)}$ vertex models can also be treated in this way. Additionally, we notice that algebraic Bethe ansatz has been generalized to the spin chain with non-diagonal reflecting matrices [24]–[27]. It is interesting to apply the method to other higher rank algebras.

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A. Derivations of diagonal operators acting on eigenvector

Acting the diagonal operators $x(u) = A(u), \tilde{A}_{aa}(u), \tilde{A}_2(u)$ on the m-particle state and having carried out a very involved analysis as that in ref. [16], we can obtain the following expression

$$\begin{aligned}
x(u)|\Upsilon_m(v_1,\ldots,v_m)\rangle &= |\Psi_x(u,\{v_m\})\rangle + \\
&+ \sum_{i=1}^{m} h_1^x(u,v_i,d_1)|\Psi_{m-1}^{(1)}(u,v_i;\{v_m\})_{d_1d_1}\rangle + \\
&+ \sum_{i=1}^{m} h_2^x(u,v_i,d)|\Psi_{m-1}^{(2)}(u,v_i;\{v_m\})_{dd}\rangle + \\
&+ \sum_{i=1}^{m} h_3^x(u,v_i,\bar{\alpha}_x)|\Psi_{m-1}^{(3)}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle + \\
&+ \sum_{i=1}^{m} h_4^x(u,v_i,\bar{\alpha}_x)|\Psi_{m-1}^{(4)}(u,v_i;\{v_m\})_{\alpha_x\alpha_x}\rangle + \\
&+ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \delta_{\bar{d}_1e_2} H_{1,d_1}^x(u,v_i,v_j)|\Psi_{m-2}^{(5)}(u,v_i,v_j;\{v_m\})_{d_1e_2}\rangle + \\
&+ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} H_{2,d_1}^x(u,v_i,v_j)_{c_2d_1}^{fe}|\Psi_{m-2}^{(6)}(u,v_i,v_j;\{v_m\})_{d_1c_2}^{ef}\rangle + \\
&+ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} H_{3,d_1}^x(u,v_i,v_j)|\Psi_{m-2}^{(7)}(u,v_i,v_j;\{v_m\})_{d_1d_1}\rangle + \\
&+ \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} H_{4,d_1}^x(u,v_i,v_j)_{fd_1}^{de}|\Psi_{m-2}^{(8)}(u,v_i,v_j;\{v_m\})_{d_1f}^{ed}\rangle\,, \qquad \text{(A.1)}
\end{aligned}$$

where when $x = A, \tilde{A}_{aa}, \tilde{A}_2$,

$$\begin{aligned}
h_1^x(u,v_i,d_1) &= a_2^1(u,v_i), R_1^A(u,v_i)_{ad_1}^{d_1a}, a_2^3(u,v_i)\,, \\
h_2^x(u,v_i,d) &= a_3^1(u,v_i), R_2^A(u,v_i)_{ad}^{da}, a_3^3(u,v_i)\,, \\
h_3^x(u,v_i,\bar{\alpha}_x) &= 0, R_3^A(u,v_i,\bar{a}), a_4^3(u,v_i,\bar{d})\,,
\end{aligned}$$

$$
\begin{aligned}
h_4^x(u,v_i,\bar{\alpha}_x) &= 0, R_4^A(u,v_i,\bar{a}), a_5^3(u,v_i,\bar{d})\,,\\
\alpha_x &= 0, a, d,\\
|\Psi_x(u,\{v_m\})\rangle &= \begin{cases} \omega_1(u)\Lambda_1^m(u;v_1,\ldots,v_m)|\Upsilon_m(v_1,\ldots,v_m)\rangle, \\ \Phi_m^{d_1\cdots d_m}(v_1,\ldots,v_m)[\tilde{T}^m(u;\{v_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}]_{aa}F^{b_1\cdots b_m}|0\rangle, \\ \omega_{2n+1}(u)\Lambda_3^m(u;v_1,\ldots,v_m)|\Upsilon_m(v_1,\ldots,v_m)\rangle, \end{cases}
\end{aligned} \tag{A.2}
$$

respectively, and we denote

$$
\begin{aligned}
|\Psi^{(1)}_{m-1}(u,v_i;\{v_m\})_{fd_1}\rangle &= B_f(u)|\Phi_{m-1}^{d_2\cdots d_m}(v_1,\ldots,\check{v}_i,\ldots,v_m)\times\\
&\quad\times S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\Lambda_1^{m-1}(v_i;\{\check{v}_i\})\omega_1(v_i)F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.3)}\\
\Psi^{(2)}_{m-1}(u,v_i;\{v_m\})_{fd}\rangle &= B_f(u)\Phi_{m-1}^{e_2\cdots e_m}(v_1,\ldots,\check{v}_i,\ldots,v_m)\times\\
&\quad\times[\tilde{T}^{m-1}(v_i;\{\check{v}_i\})^{e_2\cdots e_m}_{d'_2\cdots d'_m}]_{dd'_1}S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.4)}\\
|\Psi^{(3)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle &= E_a(u)\Phi_{m-1}^{d_2\cdots d_m}(v_1,\ldots,\check{v}_i,\ldots,v_m)\times\\
&\quad\times S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\Lambda_1^{m-1}(v_i;\{\check{v}_i\})\omega_1(v_i)\delta_{\bar{b}d_1}F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.5)}\\
|\Psi^{(4)}_{m-1}(u,v_i;\{v_m\})_{ab}\rangle &= E_a(u)\Phi_{m-1}^{e_2\cdots e_m}(v_1,\ldots,\check{v}_i,\ldots,v_m)\times\\
&\quad\times[\tilde{T}^{m-1}(v_i;\{\check{v}_i\})^{e_2\cdots e_m}_{d'_2\cdots d'_m}]_{\bar{b}d'_1}S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.6)}\\
|\Psi^{(5)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1e_2}\rangle &= F(u)\Phi_{m-2}^{e_3\cdots e_m}(v_1,\ldots,\check{v}_i,\ldots,\check{v}_j,\ldots,v_m)\times\\
&\quad\times S^{e_2\cdots e_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\Lambda_1^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})\times\\
&\quad\times\Lambda_1^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})A(v_i)A(v_j)F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.7)}\\
|\Psi^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})^{ef}_{d_1c_2}\rangle &= F(u)\Phi_{m-2}^{e_3\cdots e_m}(v_1,\ldots,\check{v}_i,\ldots,\check{v}_j,\ldots,v_m)\times\\
&\quad\times[\tilde{T}^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c_3\cdots c_m}]_{\bar{f}e}S^{c_2\cdots c_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\times\\
&\quad\times\Lambda_1^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})A(v_j)F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.8)}\\
|\Psi^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{fd_1}\rangle &= F(u)\Phi_{m-2}^{e_3\cdots e_m}(v_1,\ldots,\check{v}_i,\ldots,\check{v}_j,\ldots,v_m)\times\\
&\quad\times[\tilde{T}^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c_3\cdots c_m}]_{\bar{f}c_2}S^{c_2\cdots c_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\times\\
&\quad\times\Lambda_1^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})A(v_i)F^{b_1\cdots b_m}|0\rangle\,, && \text{(A.9)}\\
|\Psi^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})^{ed}_{d_1f}\rangle &= F(u)\Phi_{m-2}^{e_3\cdots e_m}(v_1,\ldots,\check{v}_i,\ldots,\check{v}_j,\ldots,v_m)\times\\
&\quad\times[\tilde{T}^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{a_3\cdots a_m}]_{\bar{d}e}[\tilde{T}^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{a_3\cdots a_m}_{c_3\cdots c_m}]_{fc_2}\times\\
&\quad\times S^{c_2\cdots c_m}_{d_2\cdots d_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d_1\cdots d_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})F^{b_1\cdots b_m}|0\rangle\,. && \text{(A.10)}
\end{aligned}
$$

The explicit expressions of $H^x_{l,d_1}(u,v_i,v_j), l=1,2,3,4$ are listed as below

$$
\begin{aligned}
H^A_{1,d_1}(u,v_i,v_j) &= a_4^1(u,v_i,\bar{d}_1)(c_5^1(v_i,v_j)+c_7^1(v_i,v_j))\\
&\quad+a_5^1(u,v_i)(c_4^2(v_i,v_j,\bar{d}_1)+c_6^2(v_i,v_j,\bar{d}_1))+b_2^1(u,v_i)g_1(v_i,v_j,d_1)+\\
&\quad+a_1^1(u,v_i)a_2^1(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{\bar{d}d}_{d_1\bar{d}_1}\,, && \text{(A.11)}\\
H^A_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1} &= a_4^1(u,v_i,\bar{d}_1)(c_6^1(v_i,v_j)+c_9^1(v_i,v_j))\delta_{d_1f}+\\
&\quad+a_5^1(u,v_i)(R_3^C(v_i,v_j)^{fe}_{c_2d_1}+R_4^C(v_i,v_j)^{fe}_{c_2d_1})+\\
&\quad+b_3^1(u,v_i)g_1(v_i,v_j,d_1)\delta_{d_1\bar{c}_2}+a_1^1(u,v_i)a_2^1(u,v_j)g_2(u,v_i,f)\hat{r}(v_i-u)^{fe}_{c_2d_1}\,, && \text{(A.12)}\\
H^A_{3,d_1}(u,v_i,v_j) &= a_4^1(u,v_i,\bar{d}_1)c_8^1(v_i,v_j)+a_5^1(u,v_i)c_7^2(v_i,v_j,\bar{d}_1)+\\
&\quad+b_2^1(u,v_i)g_2(v_i,v_j,d_1)+a_1^1(u,v_i)a_3^1(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{d\bar{d}}_{\bar{d}_1d_1}\,, && \text{(A.13)}\\
H^A_{4,d_1}(u,v_i,v_j)^{de}_{fd_1} &= a_4^1(u,v_i,\bar{d}_1)c_{10}^1(v_i,v_j)\delta_{d_1d}+a_5^1(u,v_i)R_5^C(v_i,v_j)^{de}_{fd_1}+\\
&\quad+b_3^1(u,v_i)g_2(v_i,v_j,d_1)\delta_{\bar{d}_1f}+a_1^1(u,v_i)a_3^1(u,v_j)g_2(u,v_i,d)\hat{r}(v_i-u)^{de}_{fd_1}\,, && \text{(A.14)}
\end{aligned}
$$

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$$
\begin{aligned}
H^{\tilde{A}_{aa}}_{1,d_1}(u,v_i,v_j) \;=\;& R^A_5(u,v_i)^{ad_1}_{d_1a}(c^1_5(v_i,v_j)+c^1_7(v_i,v_j))+b^2_2(u,v_i)g_1(v_i,v_j,d_1)+\\
&+R^A_6(u,v_i)(c^2_4(v_i,v_j,\bar{d}_1)+c^2_6(v_i,v_j,\bar{d}_1))+\tilde{r}(u+v_i)^{ad}_{da}\bar{r}(u-v_i)^{ad_1}_{d_1a}\times\\
&\times R^A_3(u,v_j,\bar{d}_1)e^1_1(v_i,u,d)+\tilde{r}(u+v_i)^{ae}_{dg}\bar{r}(u-v_i)^{gf}_{d_1a}g_1(u,v_i,h)\times\\
&\times R^A_1(u,v_j)^{d\bar{e}}_{\bar{d}_1f}\hat{r}(v_i-u)^{h\bar{h}}_{\bar{e}e}\,, && \text{(A.15)}\\
H^{\tilde{A}_{aa}}_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1} \;=\;& R^A_5(u,v_i)^{ad_1}_{d_1a}(c^1_6(v_i,v_j)+c^1_9(v_i,v_j))\delta_{d_1f}+\\
&+R^A_6(u,v_i)(R^C_3(v_i,v_j)^{fe}_{c_2d_1}+R^C_4(v_i,v_j)^{fe}_{c_2d_1})+R^F_1(u,v_i)^{fe}_{a\bar{a}}g_1(v_i,v_j,d_1)\delta_{d_1\bar{c}_2}+\\
&+\tilde{r}(u+v_i)^{ac}_{dg}\bar{r}(u-v_i)^{gh}_{d_1a}g_2(u,v_i,f)R^A_1(u,v_j)^{db}_{c_2h}\hat{r}(v_i-u)^{fe}_{bc}+\\
&+\tilde{r}(u+v_i)^{ah}_{dg}\bar{r}(u-v_i)^{g\bar{c}_2}_{d_1a}R^A_3(u,v_j,\bar{c}_2)R^{be}_3(v_i,u)^{fe}_{\bar{d}h}\,, && \text{(A.16)}\\
H^{\tilde{A}_{aa}}_{3,d_1}(u,v_i,v_j) \;=\;& R^A_5(u,v_i)^{ad_1}_{d_1a}c^1_8(v_i,v_j)+R^A_6(u,v_i)c^2_7(v_i,v_j,\bar{d}_1)+\\
&+b^2_2(u,v_i)g_2(v_i,v_j,d_1)+\tilde{r}(u+v_i)^{ad}_{da}\bar{r}(u-v_i)^{ad_1}_{d_1a}R^A_4(u,v_j,\bar{d}_1)e^1_1(v_i,u,d)+\\
&+\tilde{r}(u+v_i)^{ae}_{dg}\bar{r}(u-v_i)^{gf}_{d_1a}g_1(u,v_i,h)R^A_2(u,v_j)^{d\bar{e}}_{\bar{d}_1f}\hat{r}(v_i-u)^{h\bar{h}}_{\bar{e}e}\,, && \text{(A.17)}\\
H^{\tilde{A}_{aa}}_{4,d_1}(u,v_i,v_j)^{de}_{fd_1} \;=\;& R^A_5(u,v_i)^{ad_1}_{d_1a}c^1_{10}(v_i,v_j)\delta_{d_1d}+R^A_6(u,v_i)R^C_5(v_i,v_j)^{de}_{fd_1}+\\
&+R^F_1(u,v_i)^{de}_{a\bar{a}}g_2(v_i,v_j,d_1)\delta_{d_1\bar{f}}+\tilde{r}(u+v_i)^{ac}_{hg}\bar{r}(u-v_i)^{g\bar{f}}_{d_1a}R^A_4(u,v_j,f)R^{be}_3(v_i,u)^{de}_{\bar{h}c}+\\
&+\tilde{r}(u+v_i)^{ac}_{bg}\bar{r}(u-v_i)^{gh}_{d_1a}g_2(u,v_i,d)R^A_2(u,v_j)^{bk}_{fh}\hat{r}(v_i-u)^{de}_{kc}\,, && \text{(A.18)}\\
H^{\tilde{A}_2}_{1,d_1}(u,v_i,v_j) \;=\;& a^3_6(u,v_i,\bar{d}_1)(c^1_5(v_i,v_j)+c^1_7(v_i,v_j))+b^3_2(u,v_i)g_1(v_i,v_j,d_1)+\\
&+a^3_7(u,v_i)(c^2_4(v_i,v_j,\bar{d}_1)+c^2_6(v_i,v_j,\bar{d}_1))+a^3_1(u,v_i)a^3_4(u,v_j,\bar{d}_1)e^1_1(v_i,u,d_1)+\\
&+a^3_1(u,v_i)a^3_2(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{d\bar{d}}_{\bar{d}_1d_1}\,, && \text{(A.19)}\\
H^{\tilde{A}_2}_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1} \;=\;& a^3_6(u,v_i,\bar{d}_1)(c^1_6(v_i,v_j)+c^1_9(v_i,v_j))\delta_{d_1f}+\\
&+a^3_7(u,v_i)(R^C_3(v_i,v_j)^{fe}_{c_2d_1}+R^C_4(v_i,v_j)^{fe}_{c_2d_1})+\\
&+b^3_3(u,v_i)g_1(v_i,v_j,d_1)\delta_{d_1\bar{c}_2}+a^3_1(u,v_i)a^3_4(u,v_j,c_2)R^{be}_3(v_i,u)^{fe}_{c_2d_1}+\\
&+a^3_1(u,v_i)a^3_2(u,v_j)g_2(u,v_i,f)\hat{r}(v_i-u)^{fe}_{c_2d_1}\,, && \text{(A.20)}\\
H^{\tilde{A}_2}_{3,d_1}(u,v_i,v_j) \;=\;& a^3_6(u,v_i,\bar{d}_1)c^1_8(v_i,v_j)+a^3_7(u,v_i)c^2_7(v_i,v_j,\bar{d}_1)+\\
&+b^3_2(u,v_i)g_2(v_i,v_j,d_1)+a^3_1(u,v_i)a^3_5(u,v_j,d_1)e^1_1(v_i,u,d_1)+\\
&+a^3_1(u,v_i)a^3_3(u,v_j)g_1(u,v_i,d)\hat{r}(v_i-u)^{d\bar{d}}_{\bar{d}_1d_1}\,, && \text{(A.21)}\\
H^{\tilde{A}_2}_{4,d_1}(u,v_i,v_j)^{de}_{fd_1} \;=\;& a^3_6(u,v_i,\bar{d}_1)c^1_{10}(v_i,v_j)\delta_{d_1d}+a^3_7(u,v_i)R^C_5(v_i,v_j)^{de}_{fd_1}+\\
&+b^3_3(u,v_i)g_2(v_i,v_j,d_1)\delta_{d_1\bar{f}}+a^3_1(u,v_i)a^3_5(u,v_j,\bar{f})R^{be}_3(v_i,u)^{de}_{fd_1}+\\
&+a^3_1(u,v_i)a^3_3(u,v_j)g_2(u,v_i,d)\hat{r}(v_i-u)^{de}_{fd_1}\,. && \text{(A.22)}
\end{aligned}
$$

In eqs. (A.11)–(A.22) all the repeated indices sum over 1 to $2n-1$ except for a, d_1 and we have checked that

$$
\frac{H^x_{2,b}(u,v_i,v_j)^{d_1a}_{cb}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{d_1a}_{cb}} = \frac{H^x_{2,d_1}(u,v_i,v_j)^{d_1d_1}_{d_1d_1}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{d_1d_1}_{d_1d_1}}\,, \tag{A.23}
$$

$$
\frac{H^x_{4,b}(u,v_i,v_j)^{d_1a}_{cb}}{R^{(n-1)}(\tilde{v}_i+\tilde{v}_j)^{d_1a}_{cb}} = \frac{H^x_{4,d_1}(u,v_i,v_j)^{d_1d_1}_{d_1d_1}}{R^{(n-1)}(\tilde{v}_i+\tilde{v}_j)^{d_1d_1}_{d_1d_1}}\,. \tag{A.24}
$$

We conclude that eq. (A.1) can be verified directly by using mathematical induction, although it is a rather hard work. Similar to assumption of algebraic Bethe ansatz, we might assume that "quasi" m-particle states such as $B\Phi_{m-1}|0\rangle$, $E\Phi_{m-1}|0\rangle$, $BB\Phi_{m-2}|0\rangle$, $BE\Phi_{m-2}|0\rangle$, $EB\Phi_{m-2}|0\rangle$, $F\Phi_{m-2}|0\rangle$, $FB\Phi_{m-3}|0\rangle$ etc are linearly independent. Here all the indices are omitted and all the spectrum parameters in the "quasi" n-particle state keep the order $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ with $i_1 < i_2 < \cdots < i_k$. For example, $B_1(v_1)B_1(v_2)\Phi_{m-2}^{b_3\cdots b_m}(v_3,\ldots,v_m)F^{11b_3\cdots b_m}|0\rangle$ and $B_1(v_1)B_2(v_2)\Phi_{m-2}^{b_3\cdots b_m}(v_3,\ldots,v_m)F^{12b_3\cdots b_m}|0\rangle$ are thought to be linearly independent. Then, by using the assumption, the property of eq. (3.26) and some necessary relations, we can prove the conclusions eq. (A.1) as done in ref. [16].

In order to obtain the eigenvalue and the corresponding Bethe equations, we need to carry on the following procedure. Let

$$[\tilde{T}^m(u;\{v_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}]_{ab}F^{b_1\cdots b_m}|0\rangle = \omega(u)\Lambda_2^m(u;\{v_m\})[T^m(u;\{v_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}]_{ab}F^{b_1\cdots b_m}|0\rangle$$

$$\Lambda_2^m(u;\{v_m\}) = \prod_{i=1}^m \rho_{n-1}(u-v_i)\tilde{\rho}(u,v_i)\,, \tag{A.25}$$

$$\rho_{n-1}(u) = a_{n-1}(u)a_{n-1}(-u)\,, \qquad \tilde{\rho}(u,v) = \frac{1}{a_n(u+v)e_n(u-v)}\,, \tag{A.26}$$

$$\begin{aligned}\Big[T^m(u;\{v_m\})^{d_1\cdots d_m}_{c_1\cdots c_m}\Big]_{ab} &= \\ &= L(\tilde{u},\tilde{v}_1)^{ad_1}_{h_1g_1}L(\tilde{u},\tilde{v}_2)^{h_1d_2}_{h_2g_2}\cdots L(\tilde{u},\tilde{v}_m)^{h_{m-1}d_m}_{h_mg_m}k^-(u)_{h_m}\times \\ &\quad\times L^{-1}(-\tilde{u},\tilde{v}_m)^{h_mg_m}_{f_{m-1}c_m}L^{-1}(-\tilde{u},\tilde{v}_{m-1})^{f_{m-1}g_{m-1}}_{f_{m-2}c_{m-1}}\cdots L^{-1}(-\tilde{u},\tilde{v}_1)^{f_1g_1}_{bc_1}.\end{aligned} \tag{A.27}$$

Before deducing the eq. (3.27), we present the following four relations (the proofs are omitted here)

$$\begin{aligned}&S^{d_1\cdots d_m}_{c_1\cdots c_m}(v_i;\{\check{v}_i\})\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{c_1\cdots c_m}_{b_1\cdots b_m} = \\ &\quad= \left(\rho_{n-1}^{1/2}(0)\right)^{-1}T^{(d_1)}(v_i)\left[T^{m-1}(v_i;\{\check{v}_i\})^{d_2\cdots d_m}_{c'_2\cdots c'_m}\right]_{d_1c'_1}S^{c'_1\cdots c'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\})\,,\end{aligned} \tag{A.28}$$

$$\begin{aligned}&R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{\bar{a}_1h'_1}_{c'_2d'_1}\left[T^{m-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c'_3\cdots c'_m}\right]_{a_1h'_1}S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}) = \\ &\quad= \frac{\rho_{n-1}(\tilde{v}_i-\tilde{v}_j)\rho_{n-1}^{1/2}(0)}{\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)}\frac{R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{\bar{a}_1a_1}_{\bar{d}d}}{T^{(d)}(v_i)}\times \\ &\quad\quad\times S^{\bar{d}e_3\cdots e_m}_{h_2\cdots h_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{dh_2\cdots h_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})]\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}\,,\end{aligned} \tag{A.29}$$

$$\begin{aligned}&\left[T^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{c'_3\cdots c'_m}\right]_{\bar{c}_1c'_2}S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{c_1d'_2\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}) = \\ &\quad= \frac{\rho_{n-1}^{1/2}(0)}{\rho_{n-1}(\tilde{v}_j+\tilde{v}_i)}\frac{R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{c_1\bar{c}_1}_{e_2\bar{e}_2}R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{\bar{e}_2e_2}_{\bar{d}d}}{T^{(\bar{e}_2)}(v_j)}\times \\ &\quad\quad\times S^{\bar{d}e_3\cdots e_m}_{h_2\cdots h_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{dh_2\cdots h_m}_{d_1\cdots d_m}(v_i;\{\check{v}_i\})\}\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}\,,\end{aligned} \tag{A.30}$$

$$\begin{aligned}&R^{(n-1)}(\tilde{v}_j+\tilde{v}_i)^{\bar{a}_1h'_1}_{f_2d'_1}\left[T^{n-2}(v_i;\{\check{v}_i,\check{v}_j\})^{e_3\cdots e_m}_{a'_3\cdots a'_m}\right]_{a_1h'_1}\left[T^{m-2}(v_j;\{\check{v}_i,\check{v}_j\})^{a'_3\cdots a'_m}_{c'_3\cdots c'_m}\right]_{f_2c'_2}\times \\ &\times S^{c'_2\cdots c'_m}_{d'_2\cdots d'_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{d'_1\cdots d'_m}_{b_1\cdots b_m}(v_i;\{\check{v}_i\}) = \\ &\quad= \frac{\rho_{n-1}(\tilde{v}_i-\tilde{v}_j)\rho_{n-1}(0)}{\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)}\times \\ &\quad\quad\times\frac{R^{(n-1)}(-\tilde{v}_j-\tilde{v}_i)^{\bar{a}_1a_1}_{\bar{d}d}}{T^{(d)}(v_i)T^{(\bar{a}_1)}(v_j)}S^{\bar{d}e_3\cdots e_m}_{a_2\cdots a_m}(v_j;\{\check{v}_i,\check{v}_j\})S^{da_2\cdots a_m}_{g_1\cdots g_m}(v_i;\{\check{v}_i\})\times \\ &\quad\quad\times\tau_1(\tilde{v}_i;\{\tilde{v}_m\})^{g_1\cdots g_m}_{d_1\cdots d_m}\tau_1(\tilde{v}_j;\{\tilde{v}_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}\,,\end{aligned} \tag{A.31}$$

where

$$T^{(d_1)}(v_i) = k_d^+(v_i) R^{(n-1)}(2v_i - 4\eta)_{d_1 d}^{dd_1} \tag{A.32}$$

and one should note that all the repeated indices in eqs. (A.29), (A.30), (A.31) sum over 1 to $2n-1$ except for c_1 or a_1. We now denote

$$\begin{aligned}
|\tilde{\Psi}^{(2)}_{m-1}(u, v_i; \{v_m\})_{fd}\rangle &= \frac{\rho_{n-1}^{1/2}(0)\omega(v_i)}{T^{(d)}(v_i)} \Lambda_2^{m-1}(v_i; \{\check{v}_i\}) B_f(u) \Phi_{m-1}^{e_2\cdots e_m}(v_1, \ldots, \check{v}_i, \ldots, v_m) \times \\
&\quad \times S_{d_1\cdots d_m}^{de_2\cdots e_m}(v_i; \{\check{v}_i\}) \tau_1(\tilde{v}_i; \{\tilde{v}_m\})_{b_1\cdots b_m}^{d_1\cdots d_m} F^{b_1\cdots b_m}|0\rangle \,, \qquad (A.33) \\
|\tilde{\Psi}^{(4)}_{m-1}(u, v_i; \{v_m\})_{ab}\rangle &= \frac{\rho_{n-1}^{1/2}(0)\omega(v_i)}{T^{(\bar{b})}(v_i)} \Lambda_2^{m-1}(v_i; \{\check{v}_i\}) E_a(u) \Phi_{n-1}^{e_2\cdots e_m}(v_1, \ldots, \check{v}_i, \ldots, v_m) \times \\
&\quad \times S_{d_1\cdots d_m}^{\bar{b}e_2\cdots e_m}(v_i; \{\check{v}_i\}) \tau_1(\tilde{v}_i; \{\tilde{v}_m\})_{b_1\cdots b_m}^{d_1\cdots d_m} F^{b_1\cdots b_m}|0\rangle \,, \qquad (A.34) \\
|\tilde{\Psi}^{(5)}_{m-2}(u, v_i, v_j; \{v_m\})_{d_1}\rangle &= F(u) \Phi_{m-2}^{e_3\cdots e_m}(v_1, \ldots, \check{v}_i, \ldots, \check{v}_j, \ldots, v_m) \times \qquad (A.35) \\
&\quad \times S_{d_2\cdots d_m}^{\bar{d}_1 e_3\cdots e_m}(v_j; \{\check{v}_i, \check{v}_j\}) S_{b_1\cdots b_m}^{d_1\cdots d_m}(v_i; \{\check{v}_i\}) \times \\
&\quad \times \Lambda_1^{m-1}(v_i; \{\check{v}_i\}) \Lambda_1^{m-1}(v_j; \{\check{v}_j\}) \omega_1(v_i) \omega_1(v_j) F^{b_1\cdots b_m}|0\rangle \,, \\
|\tilde{\Psi}^{(6)}_{m-2}(u, v_i, v_j; \{v_m\})_{d_1}\rangle &= F(u) \Phi_{m-2}^{e_3\cdots e_m}(v_1, \ldots, \check{v}_i, \ldots, \check{v}_j, \ldots, v_m) \times \qquad (A.36) \\
&\quad \times S_{d_2\cdots d_m}^{\bar{d}_1 e_3\cdots e_m}(v_j; \{\check{v}_i, \check{v}_j\}) S_{c_1\cdots c_m}^{d_1\cdots d_m}(v_i; \{\check{v}_i\}) \tau_1(\tilde{v}_i; \{\tilde{v}_m\})_{b_1\cdots b_m}^{c_1\cdots c_m} \times \\
&\quad \times \Lambda_2^{m-1}(v_i; \{\check{v}_i\}) \omega(v_i) \rho_{n-1}^{\frac{1}{2}}(0) \Lambda_1^{m-1}(v_j; \{\check{v}_j\}) \omega_1(v_j) F^{b_1\cdots b_m}|0\rangle \,, \\
|\tilde{\Psi}^{(7)}_{m-2}(u, v_i, v_j; \{v_m\})_{d_1}\rangle &= F(u) \Phi_{m-2}^{e_3\cdots e_m}(v_1, \ldots, \check{v}_i, \ldots, \check{v}_j, \ldots, v_m) \times \qquad (A.37) \\
&\quad \times S_{d_2\cdots d_m}^{\bar{d}_1 e_3\cdots e_m}(v_j; \{\check{v}_i, \check{v}_j\}) S_{c_1\cdots c_m}^{d_1\cdots d_m}(v_i; \{\check{v}_i\}) \tau_1(\tilde{v}_j; \{\tilde{v}_m\})_{b_1\cdots b_m}^{c_1\cdots c_m} \times \\
&\quad \times \Lambda_2^{m-1}(v_j; \{\check{v}_j\}) \omega(v_j) \rho_{n-1}^{\frac{1}{2}}(0) \Lambda_1^{m-1}(v_i; \{\check{v}_i\}) \omega_1(v_i) F^{b_1\cdots b_m}|0\rangle \,, \\
|\tilde{\Psi}^{(8)}_{m-2}(u, v_i, v_j; \{v_m\})_{d_1}\rangle &= F(u) \Phi_{m-2}^{e_3\cdots e_m}(v_1, \ldots, \check{v}_i, \ldots, \check{v}_j, \ldots, v_m) \times \\
&\quad \times S_{d_2\cdots d_m}^{\bar{d}_1 e_3\cdots e_m}(v_j; \{\check{v}_i, \check{v}_j\}) S_{c_1\cdots c_m}^{d_1\cdots d_m}(v_i; \{\check{v}_i\}) \tau_1(\tilde{v}_i; \{\tilde{v}_m\})_{a_1\cdots a_m}^{c_1\cdots c_m} \times \\
&\quad \times \tau_1(\tilde{v}_j; \{\tilde{v}_m\})_{b_1\cdots b_m}^{a_1\cdots a_m} \Lambda_2^{m-1}(v_i; \{\check{v}_i\}) \times \\
&\quad \times \Lambda_2^{m-1}(v_j; \{\check{v}_j\}) \omega(v_i) \omega(v_j) \rho_{n-1}(0) F^{b_1\cdots b_m}|0\rangle \,. \qquad (A.38)
\end{aligned}$$

With the help of relation eq. (A.28), we can easily change the $|\Psi^{(2)}_{m-1}(u, v_i; \{v_m\})_{fd}\rangle$ and $|\Psi^{(4)}_{m-1}(u, v_i; \{v_m\})_{ab}\rangle$ into $|\tilde{\Psi}^{(2)}_{m-1}(u, v_i; \{v_m\})_{fd}\rangle$ and $|\tilde{\Psi}^{(4)}_{m-1}(u, v_i; \{v_m\})_{ab}\rangle$, respectively. It is easy to get

$$\delta_{\bar{d}_1 e_2} H^x_{1,d_1}(u, v_i, v_j) |\Psi^{(5)}_{m-2}(u, v_i, v_j; \{v_m\})_{d_1 e_2}\rangle = \tilde{H}^x_{1,d_1}(u, v_i, v_j) |\tilde{\Psi}^{(5)}_{m-2}(u, v_i, v_j; \{v_m\})_{d_1}\rangle \tag{A.39}$$

with

$$\tilde{H}^x_{1,d_1}(u, v_i, v_j) = \frac{H^x_{1,d_1}(u, v_i, v_j)}{a_1^1(v_i, v_j) a_1^1(v_j, v_i)}. \tag{A.40}$$

Using eq. (A.29) and eq. (A.23), eq. (A.30), eq. (A.31) and eq. (A.24), we can rewrite

$$H^x_{2,d_1}(u,v_i,v_j)^{fe}_{c_2d_1}|\Psi^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})^{ef}_{d_1c_2}\rangle = \tilde{H}^x_{2,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(6)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle\,, \tag{A.41}$$

$$H^x_{3,d_1}(u,v_i,v_j)|\Psi^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1d_1}\rangle = \tilde{H}^x_{3,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(7)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle\,, \tag{A.42}$$

$$H^x_{4,d_1}(u,v_i,v_j)^{de}_{fd_1}|\Psi^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})^{ed}_{d_1f}\rangle = \tilde{H}^x_{4,d_1}(u,v_i,v_j)|\tilde{\Psi}^{(8)}_{m-2}(u,v_i,v_j;\{v_m\})_{d_1}\rangle\,, \tag{A.43}$$

respectively. Where

$$\tilde{H}^x_{2,d_1}(u,v_i,v_j) = \frac{1}{\tilde{\rho}(v_i,v_j)a^1_1(v_j,v_i)}\frac{H^x_{2,d}(u,v_i,v_j)^{dd}_{dd}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{dd}_{dd}}\frac{R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{d\bar{d}}_{\bar{d}_1d_1}}{\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)T^{(d_1)}(v_i)}\,, \tag{A.44}$$

$$\tilde{H}^x_{3,d_1}(u,v_i,v_j) = \frac{H^x_{3,d}(u,v_i,v_j)R^{(n-1)}(-\tilde{v}_i-\tilde{v}_j)^{d\bar{d}}_{e\bar{e}}R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{\bar{e}e}_{\bar{d}_1d_1}}{\tilde{\rho}(v_j,v_i)a^1_1(v_i,v_j)\rho_{n-1}(\tilde{v}_j-\tilde{v}_i)\rho_{n-1}(\tilde{v}_j+\tilde{v}_i)T^{(\bar{e})}(v_j)}\,, \tag{A.45}$$

$$\begin{aligned}\tilde{H}^x_{4,d_1}(u,v_i,v_j) &= \frac{1}{\tilde{\rho}(v_i,v_j)\tilde{\rho}(v_j,v_i)\rho_{n-1}(\tilde{v}_j-\tilde{v}_i)\rho_{n-1}(\tilde{v}_i+\tilde{v}_j)}\times\\ &\quad\times\frac{H^x_{4,d}(u,v_i,v_j)^{dd}_{dd}}{R^{(n-1)}(\tilde{v}_i-\tilde{v}_j)^{dd}_{dd}}\frac{R^{(n-1)}(-\tilde{v}_j-\tilde{v}_i)^{d\bar{d}}_{\bar{d}_1d_1}}{T^{(d_1)}(v_i)T^{(d)}(v_j)}\,.\end{aligned} \tag{A.46}$$

After making the notation

$$|\tilde{\Psi}_x(u,\{v_m\})\rangle = \begin{cases}|\Psi_x(u,\{v_m\})\rangle\,, \quad x=A,\tilde{A}_2\\ \omega(u)\Lambda^m_2(u;\{v_m\})\Phi^{d_1\cdots d_m}_m(v_1,\ldots,v_m)\times\\ \times[T^m(u;\{v_m\})^{d_1\cdots d_m}_{b_1\cdots b_m}]_{aa}F^{b_1\cdots b_m}|0\rangle\,, \quad x=\tilde{A}_{aa}\end{cases} \tag{A.47}$$

we arrive at the final result eq. (3.27).

B. Necessary coefficients

Here we list all the coefficients that appear in the eigenvalue and Bethe equations.

$$\begin{aligned}\omega^{(j)}_1(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \bar{\omega}^{(j)}_1(u^{(j)})\xi^{(j)}_1(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\})\,,\\ \omega^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \bar{\omega}^{(j)}(u^{(j)})\xi^{(j)}_2(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\})\,,\\ \omega^{(j)}_{2(n-j)+1}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \bar{\omega}^{(j)}_{2(n-j)+1}(u^{(j)})\xi^{(j)}_3(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\})\end{aligned} \tag{B.1}$$

with

$$\begin{aligned}\xi^{(j)}_1(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \prod_{i=1}^{m_{j-1}}\frac{a_{n-j}(u^{(j)}+\tilde{v}^{(j-1)}_i)a_{n-j}(u^{(j)}-\tilde{v}^{(j-1)}_i)}{a_{n-j}(u^{(j)}-\tilde{v}^{(j-1)}_i)a_{n-j}(\tilde{v}^{(j-1)}_i-u^{(j)})}\,,\\ \xi^{(j)}_2(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) &= \prod_{i=1}^{m_{j-1}}\frac{b_{n-j}(u^{(j)}+\tilde{v}^{(j-1)}_i)b_{n-j}(u^{(j)}-\tilde{v}^{(j-1)}_i)}{a_{n-j}(u^{(j)}-\tilde{v}^{(j-1)}_i)a_{n-j}(\tilde{v}^{(j-1)}_i-u^{(j)})}\,,\end{aligned}$$

$$\xi_3^{(j)}(u^{(j)};\{\tilde{v}^{(j-1)}_{m_{j-1}}\}) = \prod_{i=1}^{m_{j-1}} \frac{e_{n-j}(u^{(j)}+\tilde{v}_i^{(j-1)})e_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})}{a_{n-j}(u^{(j)}-\tilde{v}_i^{(j-1)})a_{n-j}(\tilde{v}_i^{(j-1)}-u^{(j)})}\,. \tag{B.2}$$

For the case of eq. (2.11), we have

$$\begin{aligned}
\bar{\omega}_1^{(j)}(u^{(j)}) &= 1\,, \qquad \bar{\omega}^{(j)}(u^{(j)}) = \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)}\,,\\
\bar{\omega}_{2(n-j)+1}^{(j)}(u^{(j)}) &= \frac{e^{2(2(n-j)-1)\eta}\sinh(u^{(j)})\cosh(u^{(j)}-(2(n-j)+3)\eta)}{\sinh(u^{(j)}-4(n-j)\eta)\cosh(u^{(j)}-(2(n-j)+1)\eta)}\,.
\end{aligned} \tag{B.3}$$

while for the case of eq. (2.12), if $p_- = n$, we have

$$\begin{aligned}
\bar{\omega}_1^{(j)}(u^{(j)}) &= e^{-u^{(j)}}\left[c_-\cosh(\eta)+\sinh(u^{(j)}-2(n-j)\eta)\right],\\
\bar{\omega}_{2(n-j)+1}^{(j)}(u^{(j)}) &= \frac{e^{u^{(j)}-4\eta}\sinh(u^{(j)})[c_-\sinh(2\eta)+\cosh(u^{(j)}-(2(n-j)+1)\eta)]}{\sinh(u^{(j)}-4(n-j)\eta)\cosh(u^{(j)}-(2(n-j)+1)\eta)}\times\\
&\quad\times[c_-\cosh(\eta)+\sinh(u^{(j)}-2(n-j)\eta)]\,,\\
\bar{\omega}^{(j)}(u^{(j)}) &= \frac{\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)}\times\begin{cases}1, & (j\neq n-1)\\ c_-\cosh(u^{(j)}-\eta) & (j=n-1)\,.\end{cases}
\end{aligned} \tag{B.4}$$

If $p_- \neq n$, we will get

$$\begin{aligned}
\bar{\omega}_1^{(j)}(u^{(j)}) &= e^{-u^{(j)}}\left[c_-\cosh(\eta)+\sinh(u^{(j)}-2(2p_- -(n+j))\eta)\right],\\
\bar{\omega}_{2(n-j)+1}^{(j)}(u^{(j)}) &= \frac{e^{u^{(j)}-4\eta}\sinh(u^{(j)})\left[c_-\sinh(2\eta)+\cosh(u^{(j)}-(2(n-j)+1)\eta)\right]}{\sinh(u^{(j)}-4(n-j)\eta)\cosh(u^{(j)}-(2(n-j)+1)\eta)}\times\\
&\quad\times[c_-\cosh((4p_- -4(n-j)-1)\eta)+\sinh(u^{(j)}-2(n-j)\eta)]\,,\\
\bar{\omega}^{(j)}(u^{(j)}) &= \frac{\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)}
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
\bar{\omega}_1^{(j)}(u^{(j)}) &= \left[c_-\cosh(u^{(j)}-(2p_- -2j-1)\eta)+\sinh(2(n-p_-)\eta)\right],\\
\bar{\omega}_{2(n-j)+1}^{(j)}(u^{(j)}) &= \frac{e^{2(2(n-j)-1)\eta}\sinh(u^{(j)})\cosh(u^{(j)}-(2(n-j)+3)\eta)}{\sinh(u^{(j)}-4(n-j)\eta)\cosh(u^{(j)}-(2(n-j)+1)\eta)}\times\\
&\quad\times\left[c_-\cosh(u^{(j)}-(2p_- -2j-1)\eta)+\sinh(2(n-p_-)\eta)\right],\\
\bar{\omega}^{(j)}(u^{(j)}) &= \frac{e^{2\eta}\sinh(u^{(j)})}{\sinh(u^{(j)}-2\eta)}\times\\
&\quad\times\begin{cases}1 & (j\neq n-1)\\ \left[c_-\cosh(u^{(j)}-(2p_- -2j-1)\eta)+\sinh(2(n-p_-)\eta)\right] & (j=n-1)\end{cases}
\end{aligned} \tag{B.6}$$

for $j < p_-$ and $p_- \le j \le n-1$, respectively. For the case of eq. (2.13), we have

$$w_1^{(j)}\left(u^{(j)}\right) = \frac{\sinh(u^{(j)} - 2(2(n-j)+1)\eta)\cosh(u^{(j)} - (2(n-j)-1)\eta)}{\sinh(u^{(j)} - 2\eta)\cosh(u^{(j)} - (2(n-j)+1)\eta)},$$

$$w^{(j)}(u^{(j)}) = \frac{e^{-2\eta}\sinh(u^{(j)} - 2(2(n-j)+1)\eta)}{\sinh(u^{(j)} - 4(n-j)\eta)}, \quad w_{2(n-j)+1}^{(j)}(u^{(j)}) = e^{-2(2(n-j)-1)\eta}\,, \tag{B.7}$$

$$\beta_{j+1}(v_i^{(j)}) = \begin{cases} -\frac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-4(n-j)\eta)\cosh(v_i^{(j)}-(2n-2j-1)\eta)}{\sinh(v_i^{(j)}-2\eta)}, & j \le n-2 \\ -\frac{e^{2\eta}\sinh(v_i^{(n-1)})}{\sinh(v_i^{(n-1)}-2\eta)}. & j = n-1 \end{cases} \tag{B.8}$$

while for the case of eq. (2.14), if $p_+ = n$, we have

$$\begin{aligned} w_1^{(j)}(u^{(j)}) &= \frac{e^{u^{(j)}}\sinh(u^{(j)} - 2(2(n-j)+1)\eta)}{\sinh(u^{(j)} - 2\eta)\cosh(u^{(j)} - (2n-2j+1)\eta)} \times \\ &\quad \times [\sinh(2\eta) + c_+\cosh(u^{(j)} - (2(n-j)+1)\eta)] \times \\ &\quad \times [\cosh(\eta) - c_+\sinh(u^{(j)} - 2((n-j)+1)\eta)], \\ w_{2(n-j)+1}^{(j)}(u^{(j)}) &= K_+^{(2)}(u^{(j)}, n-j, n-j)_{2(n-j)+1}\,, \\ w^{(j)}(u^{(j)}) &= \frac{\sinh(u^{(j)} - 2(2(n-j)+1)\eta)}{\sinh(u^{(j)} - 4(n-j)\eta)} \times \\ &\quad \times \begin{cases} 1, & (j \ne n-1) \\ c_+\cosh\left(u^{(j)} - 5\eta\right) & (j = n-1)\,. \end{cases} \end{aligned} \tag{B.9}$$

If $p_+ \ne n$, we will get

$$\begin{aligned} w_1^{(j)}(u^{(j)}) &= \frac{e^{u^{(j)}}\sinh(u^{(j)} - 2(2(n-j)+1)\eta)}{\sinh(u^{(j)} - 2\eta)\cosh(u^{(j)} - (2n-2j+1)\eta)} \times \\ &\quad \times \left[\sinh(2\eta) + c_+\cosh(u^{(j)} - (2(n-j)+1)\eta)\right] \times \\ &\quad \times \left[\cosh((4p_+ - 4n - 1)\eta) - c_+\sinh(u^{(j)} - 2((n-j)+1)\eta)\right], \\ w_{2(n-j)+1}^{(j)}(u^{(j)}) &= K_+^{(2)}(u^{(j)}, n-j, p_+ - j)_{2(n-j)+1}\,, \\ w^{(j)}(u^{(j)}) &= \frac{\sinh(u^{(j)} - 2(2(n-j)+1)\eta)}{\sinh(u^{(j)} - 4(n-j)\eta)} \end{aligned} \tag{B.10}$$

and

$$\begin{aligned} w_1^{(j)}(u^{(j)}) &= \frac{\sinh(u^{(j)} - 2(2(n-j)+1)\eta)\cosh(u^{(j)} - (2(n-j)-1)\eta)}{\sinh(u^{(j)} - 2\eta)\cosh(u^{(j)} - (2(n-j)+1)\eta)} \times \\ &\quad \times [c_+\cosh(u^{(j)} - (4n - 2p_+ - 2j + 3)\eta) - \sinh(2(n-p_+)\eta)], \\ w_{2(n-j)+1}^{(j)}(u^{(j)}) &= e^{-4(n-j-\frac{1}{2})\eta}[c_+\cosh(u^{(j)} - (4n - 2p_+ - 2j + 3)\eta) - \sinh(2(n-p_+)\eta)]\,, \\ w^{(j)}(u^{(j)}) &= \frac{e^{-2\eta}\sinh(u^{(j)} - 2(2(n-j)+1)\eta)}{\sinh(u^{(j)} - 4(n-j)\eta)} \times \end{aligned} \tag{B.11}$$

$$\times\begin{cases}1, & (j\neq n-1)\\ [c_+\cosh(u^{(j)}-(4n-2p_+-2j+3)\eta)-\sinh(2(n-p_+)\eta)] & (j=n-1)\end{cases}$$

for $j<p_+$ and $p_+\leq j\leq n-1$, respectively. The expression of β_{j+1} is

$$\beta_{j+1}(v_i^{(j)})=\begin{cases}-\dfrac{2e^{v_i^{(j)}}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-4(n-j)\eta)}{\sinh(v_i^{(j)}-2\eta)}\times\\ \times\left[\sinh(2\eta)+c_+\cosh(v_i^{(j)}-(2n-2j+1)\eta)\right]\times\\ \times\left[\cosh((4p_+-4n-1)\eta)-c_+\sinh(v_i^{(j)}-2(n-j+1)\eta)\right],\\ (j<p_+,p_+\neq n)\\ -\dfrac{e^{v_i^{(n-1)}}\sinh(v_i^{(n-1)})}{\sinh(v_i^{(n-1)}-2\eta)}\left[\cosh(\eta)-c_+\sinh(v_i^{(n-1)}-2\eta)\right],\\ (p_+=n,j=n-1)\\ -\dfrac{2e^{2\eta}\sinh(v_i^{(j)})\sinh(v_i^{(j)}-4(n-j)\eta)\cosh(v_i^{(j)}-(2n-2j-1)\eta)}{\sinh(v_i^{(j)}-2\eta)}\times\\ \times\left[c_+\cosh(v_i^{(j)}-(4n-2p_+-2j+3)\eta)-\sinh(2(n-p_+)\eta)\right],\\ (p_+\leq j<n-1)\\ -\dfrac{e^{2\eta}\sinh(v_i^{(n-1)})}{\sinh(v_i^{(n-1)}-2\eta)}\quad(p_+\leq j=n-1)\,.\end{cases}\tag{B.12}$$

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